JOURNAL OF FUNCTIONAL ANALYSIS 76, 176–192 (1988)

Integral Kernels of the Scattering Matrices for Time-Periodic Schrödinger Equations

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Communicated by the Editors

Received February 15, 1986; revised December 17, 1986

Existence, (off-diagonal) smoothness, and energy decay of the integral kernels of scattering matrices are proved for a class of time-periodic Schrödinger equations.

1. INTRODUCTION

This is a continuation of the author's paper [9] concerning the scattering operator for time-periodic Schrödinger equations. We study here properties of the scattering operator, and in particular we prove that the scattering matrices have smooth integral kernels and that they decay as the energy jump.

We consider the Schrödinger equation with a time-periodic potential:

$$i\frac{\partial}{\partial t}\psi(t,x)=(-\Delta+V(t,x))\,\psi(t,x),\qquad \psi(t,\cdot)\in\mathscr{H}=L^2(\mathbf{R}^n), \tag{1}$$

$$V(t+\omega,x)=V(t,x),$$

where ω is the period of the potential. We suppose that V satisfies

Assumption (B). $V(t,x)$ is a real-valued C^∞-function on $\mathbf{R}\times\mathbf{R}^n$, and there is $\delta>\frac{1}{2}$ such that for any α and k,

$$|\partial_x^\alpha\,\partial_t^k\,V(t,x)|\leqslant C_{\alpha k}\langle x\rangle^{-(2\delta+|\alpha|)}.$$

We suppose $\delta<\frac{3}{2}$ for technical reasons.

In [9], we supposed, for $\beta>0$,

Assumption $(\mathrm{A})_\beta$. For some $\delta>\frac{1}{2}$ and $p>n$, $t\mapsto\langle x\rangle^{2\delta}\cdot V(t,x)$ is an $(L^p(\mathbf{R}^n)+L^\infty(\mathbf{R}^n))$-valued $C^{1+\beta}$-class function of t.

0022-1236/88 $3.00

Of course, (B) implies $(\mathrm{A})_\infty$, and all the results of [9] remain valid under Assumption (B). Under these assumptions, (1) generates a family of evolution operators $U(t, s)$, $t, s \in \mathbf{R}$, the wave operators defined by

$$W_\pm(s) = \operatorname*{s-lim}_{t \to \pm\infty} U(t, s)^{-1} \exp(-i(t-s)H_0), \qquad H_0 = -\Delta \tag{2}$$

exist and are complete,

$$\operatorname{Ran} W_\pm(s) = \mathscr{H}^{\mathrm{ac}}(U(s+\omega, s)) \qquad (s \in \mathbf{R}) \tag{3}$$

(see[11, 4, 7]), and the scattering operator defined by

$$S(s) = W_+(s)^* \, W_-(s)$$

is unitary. The main result in [9] may be stated as follows: since $V(t, x)$ is time-dependent, our system does not conserve the energy in general, and hence $S(s)$ does not conserve the kinetic energy, i.e., $[S(s), H_0] \neq 0$. Nonetheless, if the energy of the incoming wave is non-resonant, the energy jump of the outgoing wave is weak:

THEOREM 1 [9]. *Let* $(\mathrm{A})_\beta$ *be satisfied, and let the exceptional set*

$$\mathscr{E} = \left\{ \frac{2\pi}{\omega} \mu + \lambda \colon \mu \in \mathbf{Z},\ e^{-i\omega\lambda} \in \sigma_{pp}(U(s+\omega, s)) \right\} \cup \frac{2\pi}{\omega} \mathbf{Z}.$$

Suppose that J is a compact subset of $\mathbf{R}$ such that $J \cap \mathscr{E} = \varnothing$. Then for any $\varepsilon < \beta$,

$$\| P_{\{\lambda : \lambda > E\}}(H_0) \, S(s) \, P_J(H_0) \| \leqslant C E^{-(1/4 + \varepsilon)} \qquad (E > 0),$$

where$\{P_\Omega(H_0)\}$ is the spectral measure of H_0.

In this paper, we shall elaborate this theorm as follows: we denote by $\tau(\rho)$ the trace operator $\tau(\rho)\,\psi(x) = \psi(x)$ $(x \in \rho S^{n-1})$, $H^\delta(\mathbf{R}^n) \mapsto L^2(\rho S^{n-1})$ $(\delta > \frac{1}{2})$, and define $\tilde{F}(\lambda)$: $L^2_\delta(\mathbf{R}^n) \mapsto L^2(\lambda^{1/2} S^{n-1}) \equiv \mathscr{X}(\lambda)$ by $\tilde{F}(\lambda) = 2^{-1/2} \lambda^{-1/4} \tau(\lambda^{1/2}) \mathscr{F}_{x \to \xi}$ if $\lambda > 0$ and $\tilde{F}(\lambda) = 0$ if $\lambda \leqslant 0$ ($\mathscr{F}_{x \to \xi}$ denotes the Fourier transform). Then it is well known that $(\tilde{F}(\lambda), \mathscr{X}(\lambda), d\lambda)$ provides a spectral representation of H_0, i.e., $P_\Omega(H_0) = \int_\Omega \tilde{F}(\lambda)^* \, \tilde{F}(\lambda) \, d\lambda$ (Ω: a Borel set of $\mathbf{R}$) as a form on $L^2_\alpha(\mathbf{R}^n)$ $(\alpha > \frac{1}{2})$. By virtue of the time-periodicity, we have $[S(s), \exp(-i\omega H_0)] = 0$, and hence $S(s)$ has the following representation (cf. Section2 of [9]):

$$\begin{tikzcd}
\mathscr{H} \arrow[r, "\tilde{F}(\cdot)"] \arrow[d, "S(s)"'] & \int \oplus \, \mathscr{X}(\lambda) \, d\lambda \arrow[d, "\tilde{S}"] \\
\mathscr{H} \arrow[r, "\tilde{F}(\cdot)"'] & \int \oplus \, \mathscr{X}(\lambda) \, d\lambda
\end{tikzcd}$$

and

$$\tilde{S}=\sum_{\mu}\tilde{S}_{\mu}(\lambda), \qquad \tilde{S}_{\mu}(\lambda): \mathscr{X}(\lambda)\mapsto\mathscr{X}\left(\lambda-\frac{2\pi}{\omega}\mu\right).$$

We call $\{\tilde{S}_{\mu}(\lambda)\}$ scattering matrices.

THEOREM 2. *Suppose Assumption* (B) *is satisfied. Then*

(i) *for* $\lambda\notin\mathscr{E}$, $\mu\neq 0$, *there exists* $\tilde{S}_{\mu}(\lambda;\xi,\xi')\in C^{\infty}((\lambda-(2\pi/\omega)\mu)^{1/2} S^{n-1}\times\lambda^{1/2}S^{n-1})$ *such that*

$$(\tilde{S}_{\mu}(s)\psi)(\xi)=\int_{\lambda^{1/2}S^{n-1}}\tilde{S}_{\mu}(\lambda;\xi,\xi')\,\psi(\xi')\,d\xi'$$

$$\left(\psi\in\mathscr{X}(\lambda),\ \xi\in\left(\lambda-\frac{2\pi}{\omega}\mu\right)^{1/2}S^{n-1}\right), \tag{4}$$

(ii) *if* J *is a compact subset of* $\mathbf{R}$ *and* $J\cap\mathscr{E}=\varnothing$, *for any* α, β, *and* N *there is a constant* $C=C(J,\alpha,\beta,N)$ *such that*

$$|\partial_{\xi}^{\alpha}\partial_{\xi'}^{\beta}\tilde{S}_{\mu}(\lambda;\xi,\xi')|$$
$$\leqslant C\langle\mu\rangle^{-N}\left(\mu\neq 0,\ \lambda\in J,\ \xi\in\left(\lambda-\frac{2\pi}{\omega}\mu\right)^{1/2}S^{n-1},\ \xi'\in\lambda^{1/2}S^{n-1}\right), \tag{5}$$

(iii) *for any* $\lambda\in\mathbf{R}\backslash\mathscr{E}$, *there is* $\tilde{S}_0(\lambda;\xi,\xi')\in C^{\infty}(\lambda^{1/2}S^{n-1}\times\lambda^{1/2}S^{n-1}\backslash D)$ *such that* (4) *is satisfied with* $\mu=0$, *where* $D=\{(\xi,\xi):\ \xi\in\lambda^{1/2}S^{n-1}\}$.

The existence of the integral kernel of S-matrix for time-independent Schrödinger equations is well known and several proofs are available (see Reed and Simon [10], Agmon [1], Isozaki and Kitada [5, 6], and Amrein and Pearson [2]), and Theorem 2 may be considered as an extension of these theorems to time-periodic systems.

For proving Theorem 2, we first reduce the problem to that for a certain time-independent system using the Howland–Yajima method (see Section 2). Then we modify the scheme of Isozaki and Kitada [5, 6] for our case to obtain smooth kernels $\tilde{S}_{\mu}(\lambda;\xi,\xi')$: we define the time-periodic modifiers $A_{\pm}(t)$ such that they do not alter the wave operators $W_{\pm}(s)$ (Section 3) and we prove a micro-local resolvent estimate in Section 4. Using these we shall prove a representation formula for $S(s)$ in Section 5. Then Theorem 2 will be proved by combining them.

Notations. We shall use the following notations in the paper.

We denote the set of integers by $\mathbf{Z}$, natural numbers by $\mathbf{N}$, and reals by $\mathbf{R}$. We write $\mathbf{R}^m$ for the Euclidean m-space and T for the torus $\mathbf{R}/\omega\mathbf{Z}$.

For a Hilbert space $\mathscr{H}$, we write $L^p(T, \mathscr{H})$ for the $\mathscr{H}$-valued L^p-space on T. For a pair of Banach spaces X and Y, $B(X, Y)$ denotes the Banach space of all bounded operators from X to Y, and $B(X, X) = B(X)$. $L^2_\alpha(\mathbf{R}^n)$ denote the weighted L^2-space of order α on $\mathbf{R}^n$.

For a function $F = F(x)$, we often denote the operator of multiplication by $F(x)$ by the same symbol F. We write $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\hat{x} = x/|x| \in S^{n-1}$ for $x \in \mathbf{R}^n$. ∂_x^α, ∂_ξ^β, and D_x mean $(\partial/\partial x)^\alpha$, $(\partial/\partial \xi)^\beta$, and $(1/i)(\partial/\partial x)$, respectively.

$\mathscr{F}_{x \to \xi}$ denotes the Fourier transform from $\mathbf{R}^n_x$-space to $\mathbf{R}^n_\xi$-space:

$$(\mathscr{F}_{x \to \xi} \varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \exp(-ix\xi)\, \varphi(x)\, dx.$$

$\mathscr{F}_{t \to \mu} \varphi$ denotes the Fourier series expansion of φ on $T \cong [0, \omega)$:

$$(\mathscr{F}_{t \to \mu} \varphi)_\mu = \omega^{-1/2} \int_0^\omega \exp(-i2\pi\mu t/\omega)\, \varphi(t)\, dt.$$

For an operator $A \in B(L^2(T, \mathscr{H}))$, $(\mathscr{F}_{t \to \mu} A \mathscr{F}_{t \to \mu}^{-1})_{\mu\nu} \in B(\mathscr{H})$ denotes the (μ, ν)-component of $\mathscr{F}_{t \to \mu} A \mathscr{F}_{t \to \mu}^{-1} \in B(l^2(\mathscr{H}))$.

2. Preliminaries

Let $H(t)$ (H_0 resp.) be the self-adjoint operator defined on $\mathscr{H}$ by $-\Delta + V(t, x)$ ($-\Delta$ resp.) with $D(H) = D(H_0) = H^2(\mathbf{R}^n)$. Under Assumption (B), it is well known that (1) generates a family of evolution operators:

Proposition 2.1. *There exists a family of unitary operators $\{U(t, s)$: $t, s \in \mathbf{R}\}$ such that*

(i) $(t, s) \mapsto U(t, s)$: *strongly continuous*

(ii) $U(t + \omega, s + \omega) = U(t, s)$

(iii) $U(t, s)\, H^2(\mathbf{R}^n) = H^2(\mathbf{R}^n)$

(iv)
$$\frac{d}{dt} U(t, s)\psi = -iH(t)\, U(t, s)\psi$$
$$\frac{d}{dt} U(t, s)\psi = iU(t, s)\, H(s)\psi,$$

where $t, s \in \mathbf{R}$, $\psi \in H^2(\mathbf{R}^n)$ and the derivatives are taken in the strong sense in $L^2(\mathbf{R}^n)$.

For the proof, see Reed and Simon [10, Sect. X-12].

Following Yajima [11] and Howland [3, 4], we set $\mathscr{K} = L^2(T, \mathscr{H}) \cong L^2(T \times \mathbf{R}^n)$ and define self-adjoint operators K and K_0 by

$$K_0 = \frac{1}{i}\frac{\partial}{\partial t} - \Delta,$$

$$K = K_0 + V(t, x).$$

Then it is not difficult to see that they satisfy

$$(\exp(-i\sigma K_0)\psi)(t) = \exp(-i\sigma H_0)\,\psi(t-\sigma)$$

$$(\exp(-i\sigma K)\psi)(t) = U(t, t-\sigma)\,\psi(t-\sigma)$$

with $\psi = \{\psi(t)\colon t \in T,\ \psi(t) \in \mathscr{H}\} \in \mathscr{K}$, and hence

$$\begin{aligned}(\exp(i\sigma K)\exp(-i\sigma K_0)\psi)(t)&\\ &= U(t, t+\sigma)\exp(-i\sigma H_0)\,\psi(t).\end{aligned}$$

Thus the existence of the limits (2) implies that of the wave operators $\mathscr{W}_\pm$ for the pair (K, K_0), and

$$(\mathscr{W}_\pm \psi)(t) = W_\pm(t)\,\psi(t). \tag{6}$$

Moreover, (3) and (6) imply the completeness of $\mathscr{W}_\pm$ [11]:

$$\operatorname{Ran} \mathscr{W}_\pm = \mathscr{K}^{\mathrm{ac}}(K).$$

It follows that the scattering operator defined by $\mathscr{S} = \mathscr{W}_+^* \mathscr{W}_-$ is unitary. Define $F(\lambda)\colon L^2(T, L^2_\delta) \mapsto \bigoplus_{\mu < (\omega/2\pi)\lambda} \mathscr{X}(\lambda - (2\pi/\omega)\mu) \equiv \mathscr{Y}(\lambda)$ by $(F(\lambda)\Psi)_\mu = \tilde{F}(\lambda - (2\pi/\omega)\mu)\ (\mathscr{F}_{t\to\mu}\Psi)_\mu$. Then $(F(\lambda), \mathscr{Y}(\lambda), d\lambda)$ provides a spectral representation of K_0, and the scattering matrix $\tilde{\mathscr{S}}(\lambda)$: $\mathscr{Y}(\lambda) \mapsto \mathscr{Y}(\lambda)$, for the pair (K, K_0), is defined by

$$F(\lambda)\mathscr{S} = \tilde{\mathscr{S}}(\lambda)\,F(\lambda) \qquad (\lambda \in \mathbf{R}\backslash\mathscr{E}).$$

Let $\mathscr{U}_0$ and $\mathscr{U}$ be operators on $\mathscr{K}$ defined by

$$(\mathscr{U}_0\psi)(t) = \exp\{-i(t-s)H_0\}\,\psi(t) \qquad (t \in [0, \omega))$$

$$(\mathscr{U}\psi)(t) = U(t, s)\,\psi(t) \qquad (t \in [0, \omega)),$$

where $\psi \in \mathscr{K}$, s is any fixed initial time, and we have identified T with $[0, \omega)$. The next proposition follows from (6).

Proposition 2.2. *On* $\mathscr{K} \cong L^2(T) \otimes L^2(\mathbf{R}^n)$,

$$\mathscr{U}^{-1}\mathscr{W}_{\pm}\mathscr{U}_0 = 1 \otimes W_{\pm}(s)$$

$$\mathscr{U}_0^{-1}\mathscr{S}\mathscr{U}_0 = 1 \otimes S(s).$$

We prepare one more proposition on the resolvent of K:

Proposition 2.3. *Let* $(\mathrm{A})_\infty$ *be satisfied. Then, for any* $J \subset \mathbf{R}\backslash\mathscr{E}$: *compact and* $N \in \mathbf{N}$,

$$\begin{aligned}&\|(\mathscr{F}_{t\to\mu}\langle x\rangle^{-\delta}((\lambda \pm i0) - K)^{-1}\langle x\rangle^{-\delta}\mathscr{F}_{t\to\mu}^{-1})_{\mu\nu}\| \\ &\quad \leqslant C\min(\langle\mu\rangle^{N}\langle\nu\rangle^{-N}, \langle\mu\rangle^{-N}\langle\nu\rangle^{N}) \qquad (\mu, \nu \in \mathbf{Z}, \lambda \in J).\end{aligned}$$

This is a simple consequence of Proposition 3.2 (*and its proof*) *of* [9].

3. Operators $A_\pm$

Here we construct operators $A_\pm$ on $\mathscr{K}$ such that

$$T_\pm \equiv KA_\pm - A_\pm K_0 \tag{7}$$

are asumptotically small on incoming/outgoing subspaces, and that $A_\pm$ do not alter the wave operators as modifiers, i.e.,

$$\operatorname*{s-lim}_{t\to\pm\infty} \exp(i\sigma K)\, A_\pm \exp(-i\sigma K_0) = \mathscr{W}_\pm .$$

$(A_\pm f)(t) = A_\pm(t) f(t)$ $((T_\pm f)(t) = T_\pm(t) f(t)$ resp.) $(t \in T)$, and $A_\pm(t)$ $(T_\pm(t)$ resp.) are pseudo-differential operators with symbols $a_\pm(t, x, \xi)$ $(t_\pm(t, x, \xi)$ resp.):

$$(A_\pm(t)\psi)(x) = (2\pi)^{-n/2}\int \exp(ix\xi)\, a_\pm(t, x, \xi)(\mathscr{F}_{x\to\xi}\psi)(\xi)\, d\xi, \tag{8}$$

$$(T_\pm(t)\psi)(t) = (2\pi)^{-n/2}\int \exp(ix\xi)\, t_\pm(t, x, \xi)(\mathscr{F}_{x\to\xi}\psi)(\xi)\, d\xi. \tag{9}$$

Proposition 3.1. *Let* $\sigma_{1\pm}$, $\sigma_{2\pm}$, γ, *and* R *be constants so that* $-1 < \pm\sigma_{1\pm} < \pm\sigma_{2\pm} < 1$, $\gamma > 0$, *and* $R > 0$. *Then there exist* $a_\pm(t, x, \xi) = a_\pm(\sigma_{1\pm}, \sigma_{2\pm}, \gamma, R; t, x, \xi)$ *such that*

(i) *for any indices* k, α, β

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a_\pm(t, x, \xi)| \leqslant C_{k\alpha\beta}\langle\xi\rangle^{-|\beta|}\langle x\rangle^{-|\alpha|} \qquad (t\in\mathbf{R},\ x,\ \xi\in\mathbf{R}^n),$$

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta(a_\pm(t, x, \xi)-1)| \leqslant C_{k\alpha\beta}\langle\xi\rangle^{-1-|\beta|}\langle x\rangle^{-|\alpha|-(2\delta-1)} \qquad (t\in\mathbf{R},\ x,\ \xi\in\mathbf{R}^n;\ \pm\hat{x}\cdot\hat{\xi}\geqslant\sigma_{2\pm}),$$

$$a_\pm(t, x, \xi) = 0 \qquad (t\in\mathbf{R},\ x,\ \xi\in\mathbf{R}^n;\ |\xi|<\gamma \text{ or } |x|<R),$$

(ii) *the symbols* $t_\pm(t, x, \xi)$ *are given by*

$$t_\pm(t, x, \xi) = e^{-ix\xi}\left(\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x + V(t, x) - |\xi|^2\right)e^{ix\xi}a_\pm(t, x, \xi)$$
$$= \frac{1}{i}\frac{\partial}{\partial t}a_\pm + \frac{2}{i}\xi\cdot\frac{\partial}{\partial x}a_\pm - \Delta_x a_\pm + Va_\pm,$$

and they satisfy

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta t_\pm(t, x, \xi)| \leqslant C_{k\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{1-|\beta|} \qquad (t\in\mathbf{R},\ x,\ \xi\in\mathbf{R}^n),$$

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta t_\pm(t, x, \xi)| \leqslant C_{k\alpha\beta N}\langle x\rangle^{-N}\langle\xi\rangle^{1-|\beta|}$$
$$(t\in\mathbf{R},\ x,\ \xi\in\mathbf{R}^n;\ \pm\hat{x}\cdot\hat{\xi}\geqslant\sigma_{2\pm} \text{ or } \hat{x}\cdot\hat{\xi}\leqslant\sigma_{1\pm}).$$

Proof. We consider equations

$$\frac{\partial}{\partial t}a_\pm(t, x, \xi) + 2\xi\frac{\partial}{\partial x}a_\pm(t, x, \xi) - i\Delta_x a_\pm(t, x, \xi) + iV(t, x)\,a_\pm(t, x, \xi) = 0. \tag{10}$$

We construct asymptotic solutions of (10) as follows. Let $a_\pm^{(0)} = 1$, and let $a_\pm^{(m)}$ $(m\geqslant 1)$ be a solution of

$$\frac{\partial}{\partial t}a_\pm^{(m)} + 2\xi\frac{\partial}{\partial x}a_\pm^{(m)} - i\Delta_x a_\pm^{(m-1)} + iVa_\pm^{(m-1)} = 0$$

defined by

$$a_\pm^{(m)}(t, x, \xi) = i\int_0^\infty V(t\pm u, x\pm 2u\xi)\,a_\pm^{(m-1)}(t\pm u, x\pm 2u\xi, \xi)\,du$$
$$-i\int_0^\infty (\Delta_x a_\pm^{(m-1)}(t\pm u, x\pm 2u\xi, \xi))\,du$$

$$= \frac{i}{2|\xi|}\int_0^\infty V\left(t \pm \frac{u}{2|\xi|}, x \pm u\hat{\xi}\right)$$

$$\times a_\pm^{(m-1)}\left(t \pm \frac{u}{2|\xi|}, x \pm u\hat{\xi}, \xi\right) du$$

$$- \frac{i}{2|\xi|}\int_0^\infty (\Delta_x a_\pm^{(m-1)})$$

$$\times \left(t \pm \frac{u}{2|\xi|}, x \pm u\hat{\xi}, \xi\right) du.$$

They enjoy the property

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a_\pm^{(m)}(t, x, \xi)| \leqslant C_{\alpha\beta k} |\xi|^{-1-|\beta|} \langle x \rangle^{-|\alpha|-(2\delta-1)m}$$

if $\pm\hat{x}\cdot\hat{\xi} > -1+\varepsilon$, $|\xi| > \varepsilon$ with $\varepsilon > 0$, and $m \neq 0$.

Let $\chi(x)$ be a C^∞-function on $\mathbf{R}^n$ such that $\chi(x) = 0$ if $|x| \leqslant 1$, and $\chi(x) = 1$ if $|x| \geqslant 2$. Take C^∞-functions $\rho_\pm(x) = \rho_\pm(\sigma_{1\pm}, \sigma_{2\pm}; x)$ on $\mathbf{R}$ such that $\rho_\pm(x) = 0$ if $\pm x < \sigma_{1\pm}$, $\rho_\pm(x) = 1$ if $\pm x > \sigma_{2\pm}$, and $-1 \leqslant \rho_\pm(x) \leqslant 1$. Now, with a suitable choice of $\varepsilon_m \to 0$, we define $a_\pm(t, x, \xi) = a_\pm(\sigma_{1\pm}, \sigma_{2\pm}, \gamma, R; t, x, \xi)$ as

$$a_\pm(t, x, \xi) = \sum_{m=0}^{\infty} \chi(\varepsilon_m x)\, a_\pm^{(m)}(t, x, \xi)\, \chi(\xi/\gamma)\, \chi(x/R)\, \rho_\pm(\hat{x}\cdot\hat{\xi}).$$

Then Proposition 3.1 follows easily from the construction above.

By Proposition 3.1, $a_\pm(t, \cdot, \cdot) \in S^0_{1,0}$ and pseudo-differential operators $A_\pm(t) = A_\pm(\sigma_{1\pm}, \sigma_{2\pm}, \gamma, R; t)$ defined by (8) are bounded in $L^2(\mathbf{R}^n)$ (cf. Kumano-go [8]).

4. A Micro-local Resolvent Estimate

We consider pseudo-differential operators $P_\pm$ with symbols $p_\pm(t, x, \xi)$ which satisfy

Property $(P_\pm)$. $p_\pm(t, x, \xi)$ are C^∞-functions on $T \times \mathbf{R}^n \times \mathbf{R}^n$; d and $\mu_\pm$ are constants such that $d > 0$; $-1 < \mu_- < \mu_+ < 1$; and for any α, β, k, and N, there are constants $C_{k\alpha\beta}$ and $C_{Nk\alpha\beta}$ such that

$$(P_\pm - \mathrm{i}) \qquad |\partial_t^k \partial_x^\alpha \partial_\xi^\beta p_\pm(t, x, \xi)| \leqslant C_{k\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} \qquad (t \in \mathbf{R},\ x, \xi \in \mathbf{R}^n),$$

$(P_\pm - \text{ii})$ $$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta p_\pm(t, x, \xi)| \leqslant C_{Nk\alpha\beta}\langle x\rangle^{-N}\langle\xi\rangle^{-|\beta|} \qquad (t\in\mathbf{R},\ x, \xi\in\mathbf{R}^n;\ \pm\hat{x}\cdot\hat{\xi}\leqslant\mu_\pm),$$

$(P_\pm - \text{iii})$ $$p_\pm(t, x, \xi) = 0 \qquad (t\in\mathbf{R},\ x, \xi\in\mathbf{R}^n;\ |\xi|\leqslant d).$$

$P_\pm$ are operators into spaces of incoming/outgoing waves with approximately disjoint ranges.

Let $R(z)=(K-z)^{-1}$ be the resolvent of K, and let $P_\pm$ be defined by

$$(P_\pm\psi)(t, x) = p_\pm(t, x, D_x)\,\psi(t)$$

for $\psi\in\mathscr{K}$.

THEOREM 3. *Suppose $p_\pm$ satisfy $(P_\pm)$. Let $w\in\mathbf{R}$, $m\in\mathbf{Z}$, and $J\subset\mathbf{R}$ be a compact set such that $J\cap\mathscr{E}=\varnothing$. Then for any $N\in\mathbf{N}$, there is a constant C such that*

$$\|(\mathscr{F}_{t\to\mu}\langle D_x\rangle^{-1}\langle x\rangle^w P_\mp^* R(\lambda\pm i0)\, P_\pm\langle x\rangle^w\langle D_x\rangle^{-1}\mathscr{F}_{t\to\mu}^{-1})_{\mu m}\| \leqslant C\langle\mu\rangle^{-N} \qquad (u\in\mathbf{Z},\ \lambda\in J).$$

Theorem 3 is an extension of the micro-local resolvent estimate due to Isozaki and Kitada [5] to time-periodic systems. Since the proof will be similar to that in [5], we shall give only an outline.

Step 1. Set $a_\pm(t, x, \xi)=a_\pm(\sigma_{1\pm}, \sigma_{2\pm}, \gamma, 1;\ t, x, \xi)$ with $-1<\pm\sigma_{1\pm}<\pm\sigma_{2\pm}<\pm\mu_\pm$ and $\gamma^2<\text{dist}((2\pi/\omega)\mathbf{Z}, J)$. Suppose $b_\pm(t, x, \xi)$ satisfy Assumption $(P_\pm)$. Let $A_\pm$ ($B_\pm$ resp.) be the pseudo-differential operators with symbols $a_\pm$ ($b_\pm$ resp.) and let $T_\pm$ be the operators defined by (7).

Define $Q_\pm$, $U_\pm(\sigma)$, and $T_\pm(\sigma)$ by

$$\begin{aligned} Q_\pm &= A_\pm B_\pm^* \\ U_\pm(\sigma) &= A_\pm\ \exp(-i\sigma K_0) B_\pm^* \\ T_\pm(\sigma) &= T_\pm\ \exp(-i\sigma K_0) B_\pm^*. \end{aligned} \tag{11}$$

Then we see from (7) that

$$i\frac{\partial}{\partial\sigma}U_\pm(\sigma) = KU_\pm(\sigma) - T_\pm(\sigma)$$

on $C_0^\infty(T\times\mathbf{R}^n)$, and hence

$$\exp(-i\sigma K)Q_\pm = U_\pm(\sigma) - i\int_0^\sigma \exp\{-i(\sigma-\tau)K\}\, T_\pm(\tau)\,d\tau \tag{12}$$

on $\mathscr{K}$.

On the other hand, one can obtain by Propositions 3.1 and the stationary phase method (cf. Lemma 2.5 of [5]),

LEMMA 4.1. *For any w, $\gamma\geqslant 0$, and k, $N\in\mathbf{N}$, there is a constant C such that for $\pm\sigma\geqslant 0$,*

$$\|\langle x\rangle^{-(w+\gamma)}\, U_\pm(\sigma)\, \langle x\rangle^{w}\| \leqslant C\langle\sigma\rangle^{-\gamma} \tag{13}$$

$$\left\|\langle x\rangle^{-(w+\gamma)}\left[\left(\frac{\partial}{\partial t}\right)^k, U_\pm(\sigma)\right]\langle x\rangle^{w}\right\| \leqslant C\langle\sigma\rangle^{-\gamma} \tag{14}$$

$$\|\langle x\rangle^{w}\, T_\pm(\sigma)\, \langle x\rangle^{w}\, \langle D_x\rangle^{-1}\| \leqslant C\langle\sigma\rangle^{-N} \tag{15}$$

$$\left\|\langle x\rangle^{w}\left[\left(\frac{\partial}{\partial t}\right)^k, T_\pm(\sigma)\right]\langle x\rangle^{w}\,\langle D_x\rangle^{-1}\right\| \leqslant C\langle\sigma\rangle^{-N}. \tag{16}$$

For the proof, remark that $U_\pm(\sigma)$ can be expressed as

$$\begin{aligned}(U_\pm(\sigma)\psi)(t,x) = (2\pi)^{-n/2}\int a_\pm(t,x,\xi)\\ \times\exp(i\{(x-y)\xi-\sigma\xi^2\})\, b_\pm(t-\sigma,y,\xi)\,\psi(t-\sigma,y)\,dy\,d\xi.\end{aligned}$$

Then, if we define the Laplace transforms of $U_\pm(\sigma)$ and $T_\pm(\sigma)$ by

$$\hat{U}_\pm(z) = -i\int_0^{\pm\infty}\exp(i\sigma z)\, U_\pm(\sigma)\,d\sigma$$

$$\hat{T}_\pm(z) = -i\int_0^{\pm\infty}\exp(i\sigma z)\, T_\pm(\sigma)\,d\sigma$$

for z; $\pm\operatorname{Im} z>0$, (12) implies

$$R(z)Q_\pm = \hat{U}_\pm(z) - R(z)\,\hat{T}_\pm(z) \qquad (\pm\operatorname{Im} z>0), \tag{17}$$

and the next estimates follow from Lemma 4.1:

$$\|\langle x\rangle^{-(w+\gamma)}\,\hat{U}_\pm(z)\,\langle x\rangle^{w}\| \leqslant C \tag{18}$$

$$\left\|\langle x\rangle^{-(w+\gamma)}\left[\left(\frac{\partial}{\partial t}\right)^k, \hat{U}_\pm(z)\right]\langle x\rangle^{w}\right\| \leqslant C \tag{19}$$

$$\|\langle x\rangle^{w}\hat{T}_{\pm}(z)\langle x\rangle^{w}\langle D_x\rangle^{-1}\|\leqslant C \tag{20}$$

$$\left\|\langle x\rangle^{w}\left[\left(\frac{\partial}{\partial t}\right)^{k},\hat{T}_{\pm}(z)\right]\langle x\rangle^{w}\langle D_x\rangle^{-1}\right\|\leqslant C \tag{21}$$

for z: $\pm\operatorname{Im} z>0$, $\gamma>1$.

Step 2. Equations (18)–(21) imply

$$\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)}\hat{U}_{\pm}(z)\langle x\rangle^{w})\mathscr{F}^{-1}_{t\to\mu})_{kl}\|\\
&\qquad\leqslant C_N\langle k-l\rangle^{-N},\\
&\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{w}\hat{T}_{\pm}(z)\langle x\rangle^{w}\langle D_x\rangle^{-1})\mathscr{F}^{-1}_{t\to\mu})_{kl}\|\\
&\qquad\leqslant C_N\langle k-l\rangle^{-N}\qquad(\pm\operatorname{Im} z>0).
\end{aligned} \tag{22}$$

Now, by (17), we see

$$\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)}R(z)\,Q_{\pm}\langle x\rangle^{w}\langle D_x\rangle^{-1})\mathscr{F}^{-1}_{t\to\mu})_{kl}\|\\
&\quad\leqslant\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)}\hat{U}(z)\langle x\rangle^{w})\mathscr{F}^{-1}_{t\to\mu})_{kl}\|\\
&\qquad+\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)}R(z)\,\hat{T}_{\pm}(z)\\
&\qquad\times\langle x\rangle^{w}\langle D_x\rangle^{-1})\mathscr{F}^{-1}_{t\to\mu})_{kl}\|\\
&\quad\leqslant C\langle k-l\rangle^{-N}\\
&\qquad+\sum_m\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)}R(z)\langle x\rangle^{-(w+\gamma)})\mathscr{F}^{-1}_{t\to\mu})_{km}\|\\
&\qquad\times\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{(w+\gamma)}\hat{T}_{\pm}(z)\langle x\rangle^{w}\\
&\qquad\langle D_x\rangle^{-1})\mathscr{F}^{-1}_{t\to\mu})_{ml}\|\qquad(\pm\operatorname{Im} z>0).
\end{aligned} \tag{23}$$

If $w\geqslant 0$ and $\gamma>1$, the second term can be estimated by Proposition 2.3 and (22) as

(the second term in the R.H.S. of (23))

$$\begin{aligned}
&\leqslant\sum_m C_1\min(\langle k\rangle^{-N}\langle m\rangle^{N},\langle k\rangle^{N}\langle m\rangle^{-N})\,C_2\langle m-l\rangle^{-(N+2)}\\
&\leqslant C_3\min(\langle k\rangle^{-N}\langle l\rangle^{N+2},\langle k\rangle^{N}\langle l\rangle^{-(N-2)})
\end{aligned}$$

if $\operatorname{dist}(z,\mathscr{E})>\varepsilon>0$. Moreover, since $\langle x\rangle^{-(w+\gamma)}\hat{U}_{\pm}(z)\langle x\rangle^{w}$ and $\langle x\rangle^{w}\hat{T}_{\pm}(z)\langle x\rangle^{w}\langle D_x\rangle^{-1}$ are Hölder continuous in z by (13)–(16), we can take boundary values of these operators on $(\mathbf{R}\backslash\mathscr{E})\pm i0$. Thus we have proved (cf. Theorem 2.7 of [5]).

PROPOSITION 4.1. *For $w \geqslant 0$ and $\tau > 1$,*

$$\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{-(w+\gamma)} R(\lambda \pm i0)\, Q_{\pm} \langle x\rangle^{w} \langle D_x\rangle^{-1})\, \mathscr{F}_{t\to\mu}^{-1})_{ml}\| \\
&\qquad \leqslant C\langle m\rangle^{-N} \langle l\rangle^{N+2} \qquad (\lambda \in \mathbf{R}\backslash\mathscr{E}) \\
&\|(\mathscr{F}_{t\to\mu}(\langle D_x\rangle^{-1} \langle x\rangle^{w}\, Q_{\mp}^{*} R(\lambda \pm i0) \langle x\rangle^{-(w+\gamma)})\, \mathscr{F}_{t\to\mu}^{-1})_{ml}\| \\
&\qquad \leqslant C\langle m\rangle^{-N} \langle l\rangle^{N+2} \qquad (\lambda \in \mathbf{R}\backslash\mathscr{E}).
\end{aligned} \tag{24}$$

One can prove analogously to Lemma 2.10 of [5]

LEMMA 4.2. *For any w, $N > 0$ and $k \in \mathbf{N}$,*

$$\begin{aligned}
&\|\langle x\rangle^{w}\, Q_{\mp}^{*}\, U_{\pm}(\sigma) \langle x\rangle^{w}\| \leqslant C\langle\sigma\rangle^{-N} \qquad (\pm\sigma \geqslant 0), \\
&\left\|\langle x\rangle^{w}\left[\left(\frac{\partial}{\partial t}\right)^{k}, Q_{\mp}^{*}\, U_{\pm}(\sigma)\right]\langle x\rangle^{w}\right\| \leqslant C\langle\sigma\rangle^{-N} \qquad (\pm\sigma \geqslant 0).
\end{aligned} \tag{25}$$

Equation (25) implies

$$\begin{aligned}
&\left\|\langle x\rangle^{w}\left[\left(\frac{\partial}{\partial t}\right)^{k}, Q_{\mp}^{*}\, \hat{U}_{\pm}(z)\right]\langle x\rangle^{w}\right\| \\
&\qquad \leqslant C \qquad (\pm \operatorname{Im} z > 0)
\end{aligned}$$

and hence

$$\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle x\rangle^{w}\, Q_{\mp}^{*}\, \hat{U}_{\pm}(z) \langle x\rangle^{w})\, \mathscr{F}_{t\to\mu}^{-1})_{ml}\| \\
&\qquad \leqslant C\langle m-l\rangle^{-N} \qquad (\pm \operatorname{Im} z > 0).
\end{aligned}$$

Since by (17) we see

$$Q_{\mp}^{*} R(\lambda \pm i0) Q_{\pm} = Q_{\mp}^{*}\, \hat{U}_{\pm}(\lambda \pm i0) - Q_{\mp}^{*} R(\lambda \pm i0)\, \hat{T}_{\pm}(\lambda \pm i0), \tag{26}$$

combining (22), (24) with (26) we obtain

PROPOSITION 4.2. *For any $N \in \mathbf{N}$,*

$$\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle D_x\rangle^{-1} \langle x\rangle^{w}\, Q_{\mp}^{*} R(\lambda \pm i0)\, Q_{\pm} \langle x\rangle^{w} \langle D_x\rangle^{-1})\, \mathscr{F}_{t\to\mu}^{-1})_{ml}\| \\
&\qquad \leqslant C\langle m\rangle^{-N} \langle l\rangle^{N+4} \qquad (\lambda \in \mathbf{R}\backslash\mathscr{E}).
\end{aligned}$$

Step 3. Analogously to Theorem 3.4 of [5], one can prove

PROPOSITION 4.3. *Suppose $p_{\pm}(t, x, \xi)$ satisfy $(P_{\pm})$. Then for any $N \in \mathbf{N}$, there are $b_{\pm}(t, x, \xi)$ and $p_{N\pm}(t, x, \xi)$ such that*

(i) $P_{\pm} = Q_{\pm} + P_{N\pm}$,

(ii) $|\partial_t^k \partial_x^\alpha \partial_\xi^\beta p_{N\pm}(t, x, \xi)| \leqslant C_{k\alpha\beta}\langle x\rangle^{-N}$,

(iii) $b_{\pm}(t, x, \xi)$ *satisfy* $(P_{\pm})$,

where $Q_{\pm}$ *are defined by* (11).

Now, taking $N > 2w+2$ and letting $b_{\pm}$ and $p_{N\pm}$ be those in Proposition 4.3, we have

$$P_{\mp}^* R(\lambda \pm i0) P_{\pm} = Q_{\mp}^* R(\lambda \pm i0) Q_{\pm} + Q_{\mp}^* R(\lambda \pm i0) P_{N\pm} + P_{N\mp}^* R(\lambda \pm i0) Q_{\pm} + P_{N\mp}^* R(\lambda \pm i0) P_{N\pm}. \quad (27)$$

Hence Theorem 3 follows from Propositions 4.1 and 4.2 and (27).

5. Representation Theorem for $S(s)$

In this section, using $A_{\pm}$ and $T_{\pm}$, we construct a representation formula of $S(s)$ appropriate for our purpose.

Let $J \subset \mathbf{R}$ be a compact set such that $J \cap \mathscr{E} = \varnothing$. Set $d > 0$ so that $d^2 < \operatorname{dist}(J, \mathscr{E})$, and set $0 < \varepsilon < \frac{1}{4}$. Define $a_j(t, x, \xi)$ $(j = 1, 2)$ by

$$a_1(t, x, \xi) = a_+(-1+\varepsilon, -1+2\varepsilon, d, 1; t, x, \xi),$$

$$a_2(t, x, \xi) = a_+(\varepsilon, 2\varepsilon, d, 1; t, x, \xi) + a_-(-\varepsilon, -2\varepsilon, d, 1; t, x, \xi).$$

Let A_j be operators with symbols a_j and let $T_j = KA_j - A_jK_0$ $(j = 1, 2)$. Clearly, T_j are pseudo-differential operators with symbols t_j:

$$t_1(t, x, \xi) = t_+(-1+\varepsilon, -1+2\varepsilon, d, 1; t, x, \xi),$$

$$t_2(t, x, \xi) = t_+(\varepsilon, 2\varepsilon, d, 1; t, x, \xi) + t_-(-\varepsilon, -2\varepsilon, d, 1; t, x, \xi).$$

Hence the next lemma follows immediately from Proposition 3.1.

Lemma 5.1. *For any* N, k, α, β,

$$\begin{aligned} &|\partial_t^k \partial_x^\alpha \partial_\xi^\beta t_1(t, x, \xi)| \\ &\quad \leqslant C\langle x\rangle^{-N}\langle \xi\rangle^{1-|\beta|} \\ &\qquad (t \in \mathbf{R}, x, \xi \in \mathbf{R}^n; \hat{x}\cdot\hat{\xi} \in (-1, -1+\varepsilon) \cup (-1+2\varepsilon, 1)) \\ &\quad \leqslant C\langle x\rangle^{-1-|\alpha|}\langle \xi\rangle^{1-|\beta|} \qquad (\textit{otherwise}), \end{aligned}$$

$$
\begin{aligned}
&|\partial_t^k \partial_x^\alpha \partial_\xi^\beta t_2(t, x, \xi)| \\
&\quad \leqslant C\langle x\rangle^{-N} \langle\xi\rangle^{1-|\beta|} \\
&\qquad (t \in \mathbf{R}, x, \xi \in \mathbf{R}^n; \hat{x}\cdot\hat{\xi} \in (-1, -2\varepsilon) \cup (2\varepsilon, 1)) \\
&\quad \leqslant C\langle x\rangle^{-1-|\alpha|} \langle\xi\rangle^{1-|\beta|} \qquad (\textit{otherwise}).
\end{aligned}
$$

The next lemma follows from Theorem 3 and Lemma 5.1.

LEMMA 5.2. *For any w, m, and N,*

$$
\begin{aligned}
&\|(\mathscr{F}_{t\to\mu}(\langle D_x\rangle^{-2}\langle x\rangle^{w} T_1^* R(\lambda+i0) T_2 \langle x\rangle^{w}\langle D_x\rangle^{-2}) \mathscr{F}_{t\to\mu}^{-1})_{\mu m}\| \\
&\quad \leqslant C_{wN}\langle\mu\rangle^{-N} \qquad (\lambda \in J, \mu \in \mathbf{Z}).
\end{aligned}
$$

On the other hand, if we define $\mathscr{W}_{j\pm}$ by

$$
\mathscr{W}_{j\pm} = \operatorname*{s-lim}_{\sigma\to\pm\infty} \exp(i\sigma K) A_j \exp(-i\sigma K_0) E_J(K_0),
$$

one can prove the next lemma easily, using the stationary phase method.

LEMMA 5.3.

$$
\begin{aligned}
\mathscr{W}_{1+} &= \mathscr{W}_+ E_J(K_0) \\
\mathscr{W}_{2\pm} &= \mathscr{W}_\pm E_J(K_0).
\end{aligned}
$$

Remark that $(\mathscr{S}-1) E_J(K_0) = \mathscr{W}_{1+}^* (\mathscr{W}_{2+} - \mathscr{W}_{2-})$ and

$$
\mathscr{W}_{1\pm} = 1 + i\int_0^\infty \exp(i\sigma K) T_1 \exp(-i\sigma K_0) E_J(K_0)\, d\sigma,
$$

$$
\mathscr{W}_{2+} - \mathscr{W}_{2-} = i\int_{-\infty}^{+\infty} \exp(i\sigma K) T_2 \exp(-i\sigma K_0) E_J(K_0)\, d\sigma.
$$

Then, appealing to Lemma 5.2, the orthodox argument of the stationary scattering theory can be carried out to obtain (cf. Theorem 3.3 of [6])

PROPOSITION 5.1. *For $\lambda \in J$,*

$$
\begin{aligned}
\tilde{\mathscr{S}}(\lambda) = 1 &- 2\pi i F(\lambda) A_1^* T_2 F(\lambda)^* \\
&+ 2\pi i F(\lambda) T_1^* R(\lambda+i0) T_2 F(\lambda)^*.
\end{aligned}
$$

Define operators C_μ and $D_\mu(\lambda)$ by

$$
\begin{aligned}
C_\mu &= (\mathscr{F}_{t\to\mu} A_1^* T_2 \mathscr{F}_{t\to\mu}^{-1})_{\mu 0} \\
D_\mu(\lambda) &= (\mathscr{F}_{t\to\mu} T_1^* R(\lambda+i0) T_2 \mathscr{F}_{t\to\mu}^{-1})_{\mu 0}.
\end{aligned}
$$

THEOREM 4. *For $\lambda \in J$,*

$$\tilde{S}_\mu(\lambda) = \delta_{\mu 0} - 2\pi i \exp\left(is\frac{2\pi}{\omega}\mu\right) \tilde{F}\left(\lambda - \frac{2\pi}{\omega}\mu\right) C_\mu \tilde{F}(\lambda)^*$$
$$+ 2\pi i \exp\left(is\frac{2\pi}{\omega}\mu\right) \tilde{F}\left(\lambda - \frac{2\pi}{\omega}\mu\right) D_\mu(\lambda)\, \tilde{F}(\lambda)^*.$$

Theorem 4 can be proved by the same procedure as Theorem 2 of [9] from Proposition 5.1.

6. PROOF OF THEOREM 2

Clearly, it is sufficient to prove the estimate (5) for the integral kernels of $\tilde{F}(\lambda - (2\pi/\omega)\mu)\, C_\mu \tilde{F}(\lambda)^*$ and $\tilde{F}(\lambda - (2\pi/\omega)\mu)\, D_\mu(\lambda)\, \tilde{F}(\lambda)^*$, respectively.

Let $f(\xi; x) = \exp(ix\xi)$ and define $d_\mu(\lambda; \xi, \xi')$ $(\lambda \in J)$ by

$$d_\mu(\lambda; \xi, \xi') = (2\pi)^{-n}\, 2^{-1}\, |\xi|^{-1/4}\, |\xi'|^{-1/4}$$
$$\times \int \overline{f(\xi; x)}\, ((\mathscr{F}_{t\to\mu} T_1^* R(\lambda + i0)\, T_2 \mathscr{F}^{-1}_{t\to\mu})_{\mu 0}$$
$$\times f(\xi'; \cdot))(x)\, dx$$

for $\xi \in (\lambda - (2\pi/\omega)\mu)^{1/2}\, S^{n-1}$, $\xi' \in \lambda^{1/2} S^{n-1}$. This is well-defined by Lemma 5.2, and it is clear that

$$\left(\tilde{F}\left(\lambda - \frac{2\pi}{\omega}\mu\right) D_\mu(\lambda)\, \tilde{F}(\lambda)^* \psi\right)(\xi)$$
$$= \int_{\lambda^{1/2} S^{n-1}} d_\mu(\lambda; \xi, \xi')\, \psi(\xi')\, d\xi'$$

for $\lambda \in J$, $\psi \in L^2(\lambda^{1/2} S^{n-1})$, and $\xi \in (\lambda - (2\pi/\omega)\mu)^{1/2}\, S^{n-1}$. If one remarks that

$$\partial_\xi^\alpha f(\xi; x) = (ix)^\alpha f(\xi; x),$$

Lemma 5.2 implies that for any N,

$$|\partial_\xi^\alpha \partial_{\xi'}^\beta d_\mu(\lambda; \xi, \xi')|$$
$$\leqslant C\, |\xi|^{-1/4}\, |\xi'|^{-1/4} \langle \xi \rangle^2 \langle \xi' \rangle^2 C_{lN} \langle \mu \rangle^{-N} \quad (l \geqslant n/2 + \max(|\alpha|, |\beta|))$$
$$\leqslant C' \langle \mu \rangle^{-N+7/4}$$
$$(\lambda \in J,\ \mu \in \mathbf{Z},\ \xi \in (\lambda - (2\pi/\omega)\mu)^{1/2}\, S^{n-1},\ \xi' \in \lambda^{1/2} S^{n-1}).$$

Thus (5) has been proved for $\tilde{F}(\lambda - (2\pi/\omega)\mu)\, D_\mu(\lambda)\, \tilde{F}(\lambda)^*$.

On the other hand, let $c(t, \xi, \xi')$ be

$$c(t, \xi, \xi') = (2\pi)^{-n} 2^{-1} |\xi|^{-1/4} |\xi'|^{-1/4} \times \int \exp\{-ix(\xi - \xi')\} \overline{a_1(t, x, \xi)}\, t_2(t, x, \xi')\, dx$$

for $\xi, \xi' \in \mathbf{R}^n$. By Proposition 3.1, we see

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta \partial_{\xi'}^\gamma (\overline{a_1(t, x, \xi)}\, t_2(t, x, \xi'))| \leqslant C\langle x\rangle^{-1-|\alpha|} \langle\xi\rangle^{-|\beta|} \langle\xi'\rangle^{1-|\gamma|}$$

for any k, α, β, γ, and hence integration by parts shows

$$|\partial_t^k \partial_\xi^\alpha \partial_{\xi'}^\beta c(t, \xi, \xi')| \leqslant C |\xi|^{-1/4} |\xi'|^{-1/4} |\xi - \xi'|^{-(n+1)} \langle\xi\rangle^{-|\alpha|} \langle\xi'\rangle^{1-|\beta|}. \tag{28}$$

Define $c_\mu(\lambda; \xi, \xi')$ by

$$c_\mu(\lambda; \xi, \xi') = \omega^{-1} \int_0^\omega \exp(-i\mu t)\, c(t, \xi, \xi')\, dt,$$

for $\xi \in (\lambda - (2\pi/\omega)\mu)^{1/2} S^{n-1}$ and $\xi' \in \lambda^{1/2} S^{n-1}$, with $\lambda \in J$. Then it is clear from the definition that

$$\left(\tilde{F}\left(\lambda - \frac{2\pi}{\omega}\mu\right) C_\mu \tilde{F}(\lambda)\psi\right)(\xi) = \int_{\lambda^{1/2}S^{n-1}} c_\mu(\lambda; \xi, \xi')\, \psi(\xi')\, d\xi'$$

for $\lambda \in J$, $\psi \in L^2(\lambda^{1/2} S^{n-1})$, and $\xi \in (\lambda - (2\pi/\omega)\mu)^{1/2} S^{n-1}$. Moreover, (28) implies that for any $N \in \mathbf{N}$,

$$|\partial_\xi^\alpha \partial_{\xi'}^\beta c_\mu(\lambda; \xi, \xi')| \leqslant C_N \langle\mu\rangle^{-N} \qquad (\lambda \in J, \xi \in (\lambda - (2\pi/\omega)\mu)^{1/2} S^{n-1}, \xi' \in \lambda^{1/2} S^{n-1})$$

if $\mu \neq 0$. This completes the proof of (i) and (ii) of Theorem 2. Part (iii) can be proved analogously and we omit it.

ACKNOWLEDGMENTS

The author thanks Professor K. Yajima and Professor S. T. Kuroda for their encouragement and advice. He also thanks Professor B. Simon for his advice at the CIME summer school, Como, 1984.

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